ESC194 Unit 6.2

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Abstract

1 6.2

Definition: A logarithm function is a non-constant differentiable function f, defined for all real numbers between 0 and infinity, such that for all $a > 0$ and $b > 0$:

$$f(a \cdot b) = f(a) + f(b)$$

Properties:

$$f(1) = 0$$

$$f(\frac{1}{x}) = -f(x)$$

$$f(\frac{x}{y}) = f(x) - f(y))$$

$$f'(x) = \frac{1}{x} \cdot f'(1)$$

Proof:

$$f'(x) = \lim_{h->0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h->0} \frac{f(\frac{x+h}{2})}{h}$$

let $k = \frac{k}{x}$

$$= \frac{1}{x} \lim_{k->0} \frac{f(1+k) - f(1)}{h}$$

$$= \frac{1}{x} \cdot f'(x)$$

Choose $f'(1) = 1$ therefore $f'(x) = \frac{1}{x}$

$$f(x) = \int_1^x \frac{dt}{t}$$

Definition: Natural Logarithm Function

$$ln(x) = \int_1^x \frac{dt}{t} x > 0$$

Peoperties:

1) ln(x) defined on $(0, \infty)$

(ln(x))' = $\frac{1}{x}$ for $x > 0 : (ln(x))' > 0$ therefore increasing

2) ln(x) is continuous, since it's differentiable

3) for $x > 1, ln(x) > 0$

4) for $0 < x < 1, ln(x) < 0$

5) $ln(a \cdot b) = ln(a) + ln(b)$

$$ln(ax) = ln(x) + ln(a)$$

6)

$$ln(x^{\frac{p}{q}}) = \frac{p}{q} ln(x)$$

7) Range of ln(x) is $(-\infty, \infty)$

Proof of range:

$M > 0$ imposed, M very large

Show that $ln(x) > M$ for $x > x_0$

We have:

$$ln(2) = \int_1^2 \frac{dt}{t} > 0$$

Property of real numbers infinite extent:

$$nln(2) > M$$

Therefore:

$$x_0 = 2^n$$

Therefore: $ln(x) > M$ whenever $x > x_0 = 2^n$ Therefore:

$$\lim_{x \to \infty} ln(x) = \infty$$

9)

$$ln(e^{\frac{p}{q}} = \frac{p}{q})$$

10) Convention: ln(x) is log base e of x

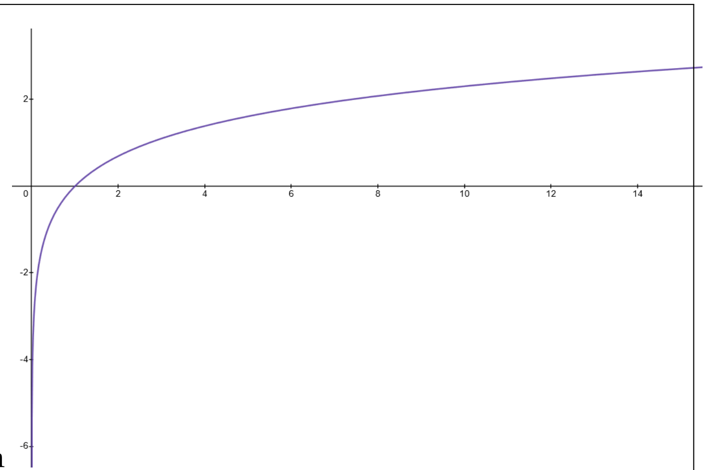
$$ln(x) = log_e(x)$$

11)

$$ln(x)' = \frac{1}{x} > 0$$

Therefore increasing

$$ln(x)'' = \frac{-1}{x^2} < 0$$



Therefore concave down

Example:

$$f(x) = ln(1 - 2x^2)$$

$$\frac{d}{dx} ln(1 - 2x^2) = \frac{-4x}{1 - 2x^2}$$

Due to domain it's required that $1 - 2x^2 > 0$ Therefore

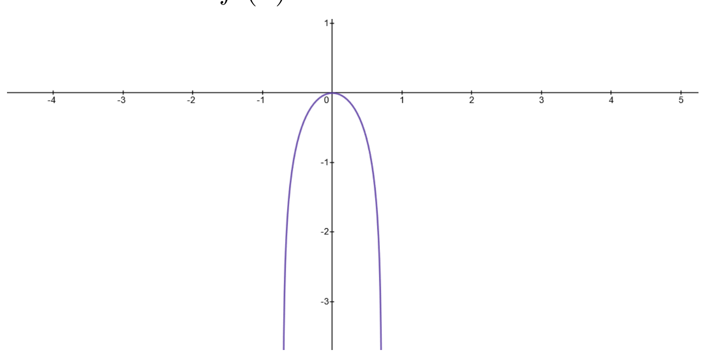
$$\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

also:

$$\lim_{x \to \pm \frac{1}{\sqrt{2}}} ln(1 - 2x^2) = -\infty$$

Therefore:

$$f'(x) = 0 \to x = 0$$



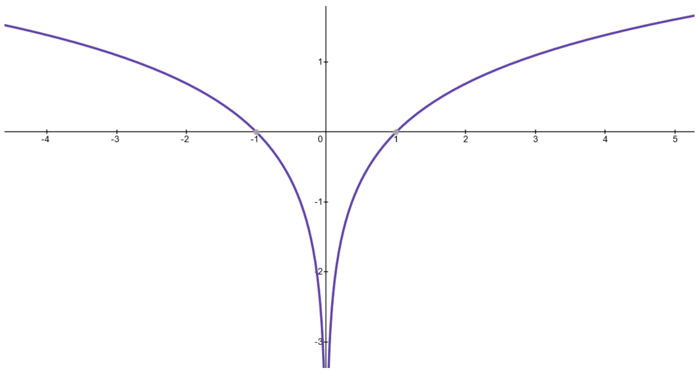
Graph:

Example:

$$f(x) = ln(|x|)$$

$$f(x) = f(-x)$$

Therefore its a horizontal reflection across y axis:



for $x > 0 ln\,|x| = lnx$

$$\frac{d}{dx} lnx = \frac{1}{x}$$

for $x < 0, ln\,|x| = ln(-x)$

$$\frac{d}{dx} ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

Chain Rule:

$$\frac{d}{dx} ln\,|u| = \frac{1}{u}\frac{du}{dx}, u \neq 0$$

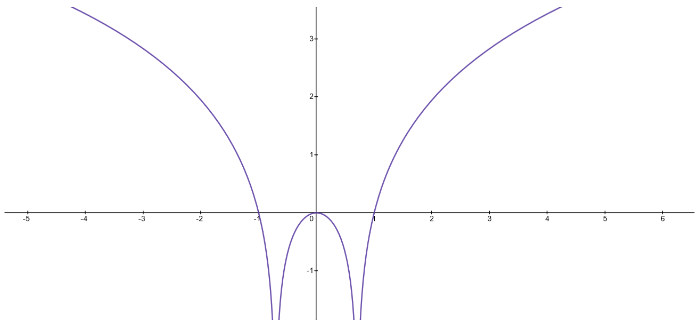
Example:

$$f(x) = ln\,\left|1 - 2x^2\right|$$

for $x^2 > \frac{1}{2}, f(x) = ln(2x^2 + 1)$

$$f'(x) = \frac{4x}{2x^2 - 1} = \frac{-4x}{1 - 2x^2}$$

as $x \rightarrow \pm\infty ln(2x^2 - 1) \rightarrow ln(2x^2) = ln(2) + 2ln\,|x| \approx 2ln(x)$ at $x = \pm 1, 2x^2 - 1 = 1$ therefore $ln2x^2 - 1 = ln(1) = 0$ Graph:



Example:

$$f(x) = \frac{1+x}{1-x}$$

Simplify:

$$ln(1+x) - ln(1-x)$$

$$f'(x) = \frac{1}{1+x} - \frac{-1}{1-x} = \frac{2}{1-x^2}$$

$$f''(x) = \frac{4x}{(1-x^2)^2}$$

Domain:

$$\frac{x+1}{x-1} > 0$$

when: $1 + x > 0$ and $1 - x > 0$
leads to: $-1 < x < 1$
when: $1 + x < 0$ and $1 - x < 0$
leads to: $x < -1$ and $x > 1$, not possible!
Therefore function lies between -1 and 1 *only*

$$f(-x) = ln(\frac{1-x}{1+x}) = -ln(\frac{1+x}{1-x}) = -f(x)$$

therefore odd function
$f' > 0$ for all x in (-1, 1), therefore increasing
$f'' < 0$ for $-1 < x < 0$ therefore concave down
$f'' > 0$ for $0 < x < 1$ therefore concave up

$f'' = 0$ when $x = 0$ means point of inflection

$$\lim_{x \to 1^-} f' = \lim_{x \to 1^+} f' = \infty$$